Exercise 4

Explain carefully the statement after Eq. A.3-17 that the il-component of $\{\boldsymbol{\sigma}\cdot\boldsymbol{\tau}\}$ is $\sum_j \sigma_{ij}\tau_{jl}$.

Solution

Taking the dot product of two second-order tensors yields a second-order tensor.

$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \boldsymbol{\eta} = \sum_{i=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_i\boldsymbol{\delta}_l\eta_{il}$$

It has two indices going from 1 to 3, meaning it has nine components in total. The coefficient of $\boldsymbol{\delta}_i\boldsymbol{\delta}_l$, η_{il}, is referred to as the il-component of $\boldsymbol{\sigma}\cdot\boldsymbol{\tau}$.

$$\begin{aligned}
\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\boldsymbol{\delta}_i\boldsymbol{\delta}_j\sigma_{ij}\right)\cdot\left(\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_k\boldsymbol{\delta}_l\tau_{kl}\right) &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_i(\boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_k)\boldsymbol{\delta}_l\sigma_{ij}\tau_{kl} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_i\delta_{jk}\boldsymbol{\delta}_l\sigma_{ij}\tau_{kl} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_i\boldsymbol{\delta}_l\sigma_{ij}\tau_{jl} \\
&= \sum_{i=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_i\boldsymbol{\delta}_l\left(\sum_{j=1}^{3}\sigma_{ij}\tau_{jl}\right)
\end{aligned}$$

Therefore, the il-component of $\boldsymbol{\sigma}\cdot\boldsymbol{\tau}$ is

$$\eta_{il} = \sum_{j=1}^{3}\sigma_{ij}\tau_{jl}.$$